Counting shifted-prime divisors

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Inspired by joint work with Carl Pomerance (Dartmouth College)

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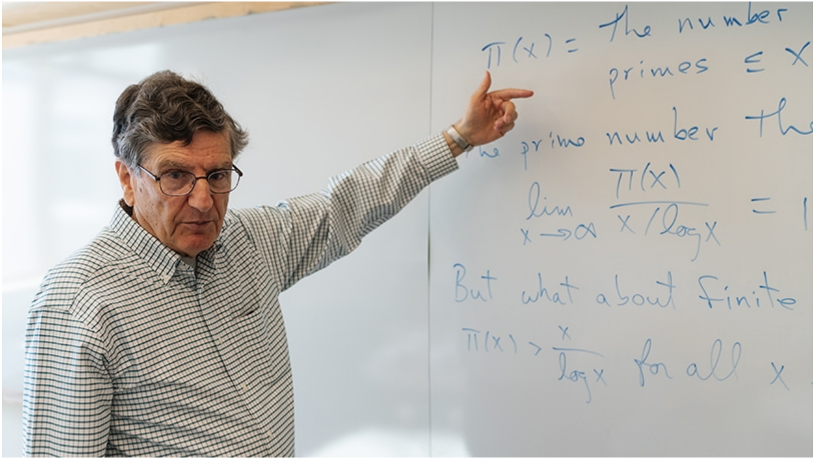


Carl Pomerance

Proof of the Sheldon Conjecture

Carl Pomerance and Chris Spicer

Abstract. In [**3**], the authors introduce the concept of a Sheldon prime, based on a conversation between several characters in the CBS television situation comedy *The Big Bang Theory*. The authors of [**3**] leave open the question of whether 73 is the unique Sheldon prime. This paper answers this question in the affirmative.

1. INTRODUCTION. A Sheldon prime was first defined in [**3**] as an homage to Sheldon Cooper, a fictional theoretical physicist, see Figure 1, on the television show *The Big Bang Theory*, who claimed 73 is the best number because it has some seemingly unusual properties. First note that not only is 73 a prime number, its index in the sequence of primes is the product of its digits, namely 21: it is the 21st prime. In addition, reversing the digits of 73, we obtain the prime 37, which is the 12th prime, and 12 is the reverse of 21.

We give a more formal definition. For a positive integer n, let p_n denote the nth prime number. We say p_n has the *product property* if the product of its base-10 digits is precisely n. For any positive integer x, we define $\operatorname{rev}(x)$ to be the integer whose sequence of base-10 digits is the reverse of the digits of x. For example, $\operatorname{rev}(1234) = 4321$ and $\operatorname{rev}(310) = 13$. We say p_n satisfies the *mirror property* if $\operatorname{rev}(p_n) = p_{\operatorname{rev}(n)}$.

Definition. The prime p_n is a Sheldon prime if it satisfies both the product property and the mirror property.

In [**3**], the "Sheldon Conjecture" was posed that 73 is the only Sheldon prime. In Section 5 we prove the following result.

Theorem 1. *The Sheldon conjecture holds: 73 is the unique Sheldon prime.*

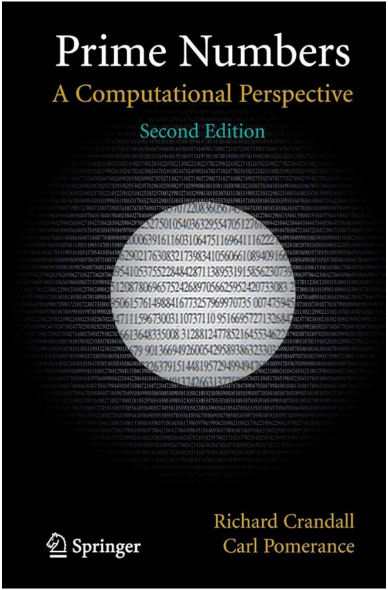
2. THE PRIME NUMBER THEOREM AND SHELDON PRIMES. Let $\pi(x)$ denote the number of prime numbers in the interval $[2, x]$. Looking at tables of primes it appears that they tend to thin out, becoming rarer as one looks at larger numbers. This can be expressed rigorously by the claim that $\lim_{x\to\infty} \pi(x)/x = 0$. In fact, more is true: we know the rate at which the ratio $\pi(x)/x$ tends to 0. This is the prime number theorem:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1,$$

where "log" is the natural logarithm function. This theorem was first proved in 1896 independently by Hadamard and de la Vallée Poussin, following a general plan laid out by Riemann about 40 years earlier (the same paper where he first enunciated the now famous Riemann hypothesis).

We actually know that $\pi(x)$ is slightly larger than $x/\log x$ for large values of x; in fact there is a secondary term $x/(\log x)^2$, a positive tertiary term, and so on. The phrase "large values of x" can be made numerically explicit: A result of Rosser and Schoenfeld [**7**, (3.5)] is that

$$\pi(x) > \frac{x}{\log x} \text{ for all } x \geq 17. \tag{1}$$



Shifted primes

A *shifted prime* is an integer of the form $p - a$, where p is prime and $a \in \mathbb{Z} \setminus \{0\}$.

We say that $p - a \neq 0$ is a *shifted-prime divisor* of $n \in \mathbb{N}$ if $(p - a) \mid n$.

For each $n \in \mathbb{N}$, denote by $\omega_a^*(n)$ the number of shifted-prime divisors $p - a$ of n:

$$\omega_a^*(n) := \#\{p > a \text{ prime} \colon (p - a) \mid n\}.$$

We will focus mainly on $\omega^*(n) := \omega_1^*(n)$ and visit briefly the general case near the end of the talk.

Example

Shifted-prime divisors $p - 1$ of 24: 1, 2, 4, 6, 12. So $\omega^*(24) = 5$.

The set $\mathbb{P}_a$

The sets $\mathbb{N}$ and $\mathbb{P}_a = \{p - a \colon p > a\}$ are structurally similar in many ways.

- Equidistribution in arithmetic progressions:

$$\frac{1}{x} \cdot \#\{n \in \mathbb{N} \cap [1, x] \colon n \equiv b \pmod{k}\} \sim \frac{1}{k},$$
$$\frac{1}{\pi(x)} \cdot \#\{n \in \mathbb{P}_a \cap [1, x] \colon n \equiv c \pmod{k}\} \sim \frac{1}{\varphi(k)},$$

as $x \to \infty$, where $k \in \mathbb{N}$, $b \in \mathbb{Z}$, and $c \in \mathbb{Z}$ with $\gcd(a + c, k) = 1$.

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  as $x \to \infty$, where $k \in \mathbb{N}$, $b \in \mathbb{Z}$, and $c \in \mathbb{Z}$ with $\gcd(a + c, k) = 1$.

- Typical number of prime factors: Let $S = \mathbb{N}$ or $\mathbb{P}_a$, and $S_x = S \cap [1, x]$.

  Most numbers n in S have about $\log\log n$ prime factors.

$$\lim_{x \to \infty} \frac{1}{\#S_x} \cdot \#\left\{n \in S_x : \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \le T\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{T} e^{-t^2/2}\, dt.$$

  $S = \mathbb{N}$ due to Erdős and Kac (1940) and $S = \mathbb{P}_a$ due to Halberstam (1955).

The set $\mathbb{P}_a$

- Twin primes: Are there infinitely many shifted primes $p+2$ that are prime?

  The recent breakthroughs made by Zhang, Maynard and Polymath, building on early works of Goldston, Pintz and Yıldırım, shows that there exists an even integer $2 \leq a \leq 246$ such that there are infinitely many shifted primes $p+a$ that are prime.

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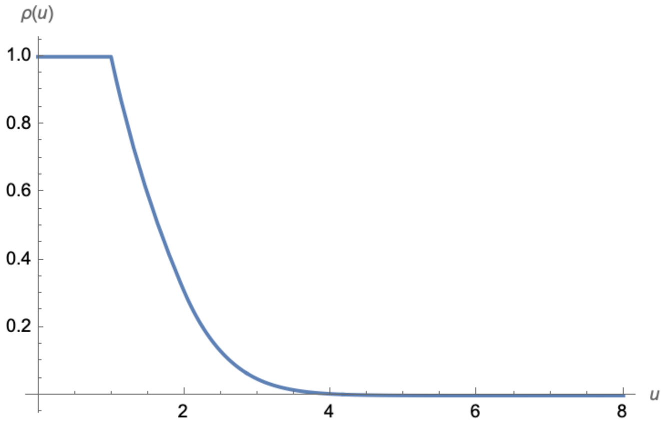
- A conjecture of Pomerance on smooth shifted primes (1980):

$$\frac{1}{\pi(x)} \cdot \# \left\{p \leq x \colon P^+(p-1) \leq y\right\} \sim \frac{1}{x} \cdot \# \left\{n \leq x \colon P^+(n) \leq y\right\}$$

  for $x \geq y$ as $y \to \infty$, where $P^+(m)$ is the largest prime factor of m.

Density of smooth numbers

Figure 1: The Dickman–de Bruijn function $\rho(u)$ on $[1, 8]$



$$\frac{1}{x} \cdot \#\{n \leq x \colon P^+(n) \leq x^{1/u}\} \approx \rho(u).$$

The function ω^*

The function

$$\omega^*(n) := \sum_{(p-1)|n} 1$$

was first introduced by Prachar (1955). It has played important roles in

- the 1983 development of the first unconditional deterministic primality test, running in nearly polynomial time, by Adleman, Pomerance and Rumely,
- the study of Carmichael numbers:

  A Carmichael number n is a composite number satisfying $b^n \equiv b \pmod{n}$ for all $b \in \mathbb{Z}$. Korselt showed in 1899 that $n \in \mathbb{N}$ is a Carmichael number if and only if n is square-free, and $p \mid n \Rightarrow p - 1 \mid n - 1$. Alford, Granville and Pomerance (1994) proved that for sufficiently large x, the interval $[1, x]$ contains at least $x^{2/7}$ Carmichael numbers. The exponent "$2/7$" has been improved to 0.332 by Harman (2005) and to 0.3389 by Lichtman (2022).

The maximal order of ω^*

The minimal order of ω^* is trivially 1: $\omega^*(n) = 1$ for odd $n \in \mathbb{N}$.

For the maximal orders, we have

$$\limsup_{x\to\infty} \frac{\omega(n)}{\log n/\log\log n} = 1,$$
$$\limsup_{x\to\infty} \frac{\log\tau(n)}{\log n/\log\log n} = \log 2. \quad \text{(Wigert, 1907)}$$

Prachar (1955) showed that for infinitely many n,

$$\omega^*(n) > \exp\left(c_1 \frac{\log n}{(\log\log n)^2}\right) \quad \text{(unconditionally)},$$
$$\omega^*(n) > \exp\left(\left(\frac{1}{2}\log 2 - \epsilon\right)\frac{\log n}{\log\log n}\right) \quad \text{(under GRH)},$$

where $c_1 > 0$ is some absolute constant, and $\epsilon > 0$ is fixed but otherwise arbitrary.

The maximal order of ω^*

Adleman, Pomerance and Rumely (1983) removed one $\log\log n$ factor from Prachar's unconditional bound, obtaining

$$\omega^*(n) > \exp\left(c_2 \frac{\log n}{\log\log n}\right)$$

for infinitely many n, where $c_2 > 0$ is some absolute computable constant.

They also conjectured that one can take $c_2 = \log 2 - \epsilon$ for any $\epsilon > 0$. This conjecture, if true, would imply that the minimal order of the Carmichael function $\lambda(n) := \mathrm{Exp}(\mathbb{Z}/n\mathbb{Z})^\times$ is

$$\exp\left(\frac{1}{\log 2}(\log\log n)\log\log\log n\right),$$

as indicated in Erdős, Pomerance, and Schmutz (1991).

The maximal order of ω^*

Recently, Pollack and I examined Prachar's proof of his GRH-conditional estimate and observed the following:

1. The Adleman–Pomerance–Rumely conjecture holds if given any $\epsilon \in (0,1)$,

$$\pi(x;q,1) := \#\{p \le x : p \equiv 1 \pmod q\} \gg_\epsilon \frac{\pi(x)}{\varphi(q)}$$

for $q \mid \prod_{p \le (1-\epsilon)\log x} p$ possibly with extremely rare exceptions.

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for $q \mid \prod_{p \le (1-\epsilon)\log x} p$ possibly with extremely rare exceptions.
2. The Adleman–Pomerance–Rumely conjecture also follows from Pomerance's conjecture that
$$\frac{1}{\pi(x)} \cdot \#\left\{p \le x : P^+(p-1) \le y\right\} \sim \frac{1}{x} \cdot \#\left\{n \le x : P^+(n) \le y\right\}$$
for $x \ge y$ as $y \to \infty$. It is directly related to lower bounding
$$\sum_{p \le x} \prod_{\substack{q \le (1-\epsilon)\log x \\ q \mid (p-1)}} q.$$

The maximal order of ω^*

3. By modifying Prachar's argument, we found that for infinitely many n,

$$\omega^*(n) > \exp\left(0.6269 \ln 2 \cdot \frac{\log n}{\log\log n}\right) \quad \text{(unconditionally)},$$
$$\omega^*(n) > \exp\left(0.6823 \ln 2 \cdot \frac{\log n}{\log\log n}\right) \quad \text{(under GRH)}.$$

The first inequality is derived from a result on $\pi(x; q, a)$ in Alford, Granville and Pomerance (1994).

The densities $\delta_k(\omega^*)$

For any arithmetic function f, we denote by $\delta_k(f)$ the *natural density* of the level set $\{n \in \mathbb{N} \colon f(n) = k\}$ for each $k \in \mathbb{N}$, namely,

$$\delta_k(f) := \lim_{x \to \infty} \frac{\#\{n \le x \colon f(n) = k\}}{x},$$

provided that this limit exists.

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$$\delta_k(f) := \lim_{x \to \infty} \frac{\#\{n \le x \colon f(n) = k\}}{x},$$

provided that this limit exists. Landau (1900) showed that for every fixed $k \in \mathbb{N}$,

$$\#\{n \le x \colon \omega(n) = k\} \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x}$$

as $x \to \infty$. So $\delta_k(\omega) = 0$. Since $\tau(n) \ge 2^{\omega(n)}$, we also have $\delta_k(\tau) = 0$ for every $k \in \mathbb{N}$.

What about $\delta_k(\omega^*)$?

The densities $\delta_k(\omega^*)$

Theorem 1 (F.–Pomerance, 2024)

For every $k \in \mathbb{N}$, the k-level set $\mathcal{L}_k := \{n \in \mathbb{N} \colon \omega^*(n) = k\}$ admits a positive natural density δ_k. Moreover, we have $\sum_{k\geq 1} \delta_k = 1$.

The key step in establishing Theorem 1 is to verify $\mathcal{L}_k \neq \emptyset$. The proof makes use of Chen's theorem:

$$\mathcal{P}_2(x) := \{2 < p \leq x \colon \Omega((p-1)/2) \leq 2 \text{ and } P^-((p-1)/2) > x^{3/11}\} \gg \frac{x}{(\log x)^2}.$$

Fixing $n \in 2\mathbb{N}$, we wish to find some large $p \in \mathcal{P}_2(x)$ such that

$$\omega^*(n(p-1)/2) = \omega^*(n) + 1.$$

If $p \in \mathcal{P}_2(x)$ fails this property, then there are $a \mid n$ and $b \mid (p-1)/2$ with $a, b > 1$ such that $ab + 1$ a prime $\neq p$.

1. $b = (p-1)/2$ with $ab + 1$ is prime with $a > 2$
2. $p - 1 = 2qb$ with $ab + 1$ prime, where $q, r \in (x^{3/11}, x^{8/11})$.

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2. $p - 1 = 2qb$ with $aq+1$ prime, where $q, r \in (x^{3/11}, x^{8/11})$.

   For any $q \in (x^{3/11}, x^{8/11})$, the number of primes $b < x/2q$ such that both $ab+1$ and $2qb+1$ are prime is

$$\ll \frac{x}{q(\log x)^3} \prod_{r|(2q-a)} \left(1 - \frac{1}{r}\right)^{-1} \ll \frac{\log\log q}{q} \cdot \frac{x}{(\log x)^3}.$$

   Summing this bound on $q \in (x^{3/11}, x^{8/11})$ gives $\ll x \log\log x/(\log x)^3$ for the number of p in consideration.

The contributions from both cases are $o(\#\mathcal{P}_2(x))$. So for sufficiently large x, we can find $p \in \mathcal{P}_2(x)$ satisfying

$$\omega^*(n(p-1)/2) = \omega^*(n) + 1.$$

The densities $\delta_k(\omega^*)$

Table 1: Exact counts of level sets for $k < 12$

k	10^4	10^6	10^8	10^{10}	$\approx \delta_k$
1	5,000	500,000	50,000,000	5,000,000,000	.5
2	834	77,696	7,436,825	720,726,912	.070
3	965	91,602	8,826,498	859,002,140	.084
4	877	79,986	7,691,971	748,412,490	.074
5	612	59,518	5,684,323	555,900,984	.055
6	456	40,641	4,031,009	401,146,301	.040
7	287	29,565	3,016,881	300,330,932	.030
8	202	23,190	2,324,769	233,611,502	.023
9	153	17,914	1,800,298	182,793,491	.018
10	159	13,899	1,401,307	144,740,573	.015
11	103	10,487	1,131,836	118,302,267	.012
≥ 12	352	55,682	6,654,283	735,032,408	

The largest values of k encountered here up to the various bounds: 10^4: 28, 10^6: 86, 10^8: 247, 10^{10}: 618. Perhaps the densities δ_k are monotone for $k \geq 3$.

The densities $\delta_k(\omega^*)$

We have seen that $\delta_k(\omega) = \delta_k(\tau) = 0$ for every fixed $k \in \mathbb{N}$. Consequently, the densities of the tails $\{n \in \mathbb{N} \colon \omega(n) > k\}$ and $\{n \in \mathbb{N} \colon \tau(n) > k\}$ are both equal to 1.

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Theorem 2 (F.–Pomerance, 2024)

For $x, y \geq 1$, let $N(x,y) := \#\{n \leq x \colon \omega^*(n) \geq y\}$. Then there exists a suitable constant $c > 0$ such that for all $x \geq 1$ and all sufficiently large y,

$$\left\lfloor \frac{x}{y^{c \log\log y}} \right\rfloor \leq N(x,y) \ll \frac{x \log y}{y}.$$

The lower bound follows from the result of Adleman, Pomerance and Rumely (1983) on the maximal order of ω^*, while the proof of the upper bound makes use of a theorem due to McNew, Pollack and Pomerance (2017), which asserts that the number of $n \leq x$ with a shifted prime divisor $> y$ is $O(x/(\log y)^{\eta+o(1)})$, where $\eta = 1 - (1 + \log\log 2)/\log 2$ is the Erdős–Ford–Tenenbaum constant.

Bernoulli numbers sharing the same fractional part

In our proof of Theorem 1, we used a result of Erdős and Wagstaff (1980) concerning the density $\delta(\langle n\rangle)$ of $\langle n\rangle$ for a given $n \in \mathbb{N}$, where

$$\begin{aligned}\langle n\rangle &:= \#\{m \in \mathbb{N}\colon B_m \equiv B_n \pmod 1\}\\ &= \#\{m \in \mathbb{N}\colon (p-1) \mid m \Leftrightarrow (p-1) \mid n\}. \quad \text{(von Staudt–Clausen)}\end{aligned}$$

Note that $\langle 1\rangle = \mathcal{L}_1 = \mathbb{N} \setminus 2\mathbb{N}$, so that $\delta(\langle n\rangle) = 1/2$ for odd n. Erdős and Wagstaff showed that $\delta(\langle n\rangle)$ exists and is positive for every $n \in \mathbb{N}$. They also observed that if $n = \min\langle n\rangle$, then $\delta(\langle n\rangle) < 1/n$. In this case, they asked for a positive lower bound for $\delta(\langle n\rangle)$.

Theorem 3 (F.–Pomerance, 2024)

Let $n \in 2\mathbb{N}$ *be such that* $n = \min\langle n\rangle$. *Then*

$$\delta(\langle n\rangle) \geq \frac{1}{n^{O(\tau(n))}}.$$

The moments of ω^*

For any arithmetic function f, we denote by $M_k(x; f)$ the *kth moment* of f for each $k \in \mathbb{N}$. That is,

$$M_k(x; f) := \frac{1}{x} \sum_{n \le x} f(n)^k.$$

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For every fixed $k \in \mathbb{N}$, we have

$$M_k(x; \omega) \sim (\log \log x)^k,$$
$$M_k(x; \tau) \sim a_k (\log x)^{2^k - 1},$$

where

$$a_k := \frac{1}{(2^k - 1)!} \prod_p \left(1 - \frac{1}{p}\right)^{2^k} \sum_{\nu \geq 0} \frac{(\nu + 1)^k}{p^\nu}.$$

The moments of ω^*

Prachar (1955) showed $M_1(x;\omega^*) \sim \log\log x$, by observing that

$$\sum_{n\le x}\omega^*(n) = \sum_{n\le x}\sum_{p-1|n} 1 = \sum_{p\le x+1}\left\lfloor \frac{x}{p-1}\right\rfloor = x\sum_{p\le x}\frac{1}{p-1} + O\left(\frac{x}{\log x}\right).$$

He also proved $M_2(x;\omega^*) = O((\log x)^2)$, which was improved to $O(\log x)$ by Murty and Murty (2021), who also showed $M_2(x;\omega^*) \gg (\log\log x)^3$ and conjectured $M_2(x;\omega^*) \sim C\log x$ for some constant $C > 0$. Ding (2023) obtained the order matching lower bound $M_2(x;\omega^*) \gg \log x$.

In general,

$$M_k(x;\omega^*) = \frac{1}{x}\sum_{n\le x}\omega^*(n)^k = \frac{1}{x}\sum_{[p_1-1,\ldots,p_k-1]\le x}\left\lfloor \frac{x}{[p_1-1,\ldots,p_k-1]}\right\rfloor.$$

The moments of ω^*

Erdős and Prachar (1955) showed

$$S_2(x) := \sum_{[p-1,q-1]\le x} 1 = O(x),$$

which allows us to write

$$M_2(x;\omega^*) = \frac{1}{x}\sum_{[p-1,q-1]\le x}\left\lfloor \frac{x}{[p-1,q-1]}\right\rfloor = \sum_{[p-1,q-1]\le x}\frac{1}{[p-1,q-1]} + O(1).$$

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Theorem 4 (F.–Pomerance, 2024)

We have $M_3(x;\omega^*) \asymp (\log x)^4$ *for all* $x \ge 2$.

Conjecture 1 (F.–Pomerance, 2024)

For every $k \ge 2$, $M_k(x;\omega^*) \sim C_k(\log x)^{2^k-k-1}$, *where* $C_k > 0$ *is constant.*

The constant C_2

Under the Elliott–Halberstam conjecture, Ding, Guo, and Zhang (2023) deduced that $C_2 = 2\zeta(2)\zeta(3)/\zeta(6) \approx 3.88719$. However, an error found in their paper (inherited from Murty and Murty (2021)) by Pomerance and I shows that this value is probably incorrect.

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Moreover, numerical computations seem to suggest $C_2 \approx 3.2$.

Conjecture 2 (F., 2025)

We have

$$C_2 = \frac{\zeta(2)^2\zeta(3)}{\zeta(6)} \approx 3.19709.$$

This conjecture is suggested by a heuristic based on the Hardy–Littlewood conjecture on the infinitude of prime pairs $p = an + 1$ and $q = bn + 1$, where $1 \leq a < b$.

The constant C_2

Table 2: Numerical values of $M_2(10^k;\omega^*)$ and $S_2(10^k;\omega^*)$

k	$M_2(10^k)$	$S_2(10^k)$	$A(10^k)$	$B(10^k)$
2	9.71	2.42	9.34061	2.5028
3	15.530	2.624	15.4058	2.7342
4	21.9128	2.8175	21.8477	2.8499
5	28.49311	2.88636	28.4958	2.9193
6	35.261891	2.950910	35.2745	2.9656
7	42.1296839	2.9923851	42.1432	2.9987
8	49.07181351	3.02166709	49.0779	3.0235
9	56.067311859	3.043042188	56.0629	3.0428
10	63.1033824202	3.0595625181	63.0876	3.0582

The M_2 values fits nicely with $A(x) := C_2(\log x - \log\log x) - 1/2$, and the S_2 values may fit with $B(x) := C_2(1 - 1/\log x)$.

The shifted-prime divisor function over shifted primes

Theorem 5 (F., 2025)

For any fixed $a, b \in \mathbb{Z} \setminus \{0\}$, we have

$$\frac{1}{\pi(x)} \sum_{b<q\leq x} \omega_a^*(q-b) = C_{a,b} \log\log x + O\left(1\right),$$

where $C_{a,b} \geq 0$ is an explicit constant.

Theorem 6 (F., 2025)

For any fixed $a, b \in \mathbb{Z} \setminus \{0\}$ such that $2 \mid a$ or $2 \nmid b$, we have

$$\frac{1}{\pi(x)} \sum_{b<q\leq x} \omega_a^*(q-b)^2 \asymp \log x.$$

A generalization of Erdős–Prachar

The treatment of the second moment requires upper bounding

$$\sum_{\substack{[p-a,q-b]\le x\\ p>a,q>b}} f([p-a,q-b]),$$

which may be viewed as a two-dimensional analogue of

$$\sum_{a<p\le x+a} f(p-a),$$

where $f:\mathbb{N}\to\mathbb{R}_{\ge 0}$ is a "nice" multiplicative function.

Erdős and Prachar (1955) showed

$$\sum_{[p-1,q-1]\le x} 1 = O(x).$$

A generalization of Erdős–Prachar

For any $A_1 > 0$ and $A_2 \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, denote by $\mathscr{M}(A_1, A_2)$ the collection of multiplicative functions $f \colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ satisfying:

1. $f(n) \leq A_1^{\Omega(n)}$ for all $n \in \mathbb{N}$.
2. $\forall \epsilon > 0$, $f(n) \leq A_2(\epsilon) n^{\epsilon}$ for all $n \in \mathbb{N}$.

Theorem 7 (F., 2025)

Let $a, b \in \mathbb{Z} \setminus \{0\}$, $A_1 > 0$, $A_2 \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, *and* $f \in \mathscr{M}(A_1, A_2)$. *Then*

$$\sum_{\substack{[p-a, q-b] \leq x \\ p > a, q > b}} f([p-a, q-b]) \ll_{a,b,A_1,A_2} \frac{x}{(\log x)^2} E_f(x) \int_2^x \frac{E_f(t)^2}{t(\log t)^2}\, dt$$

for all $x \geq 2$, *where*

$$E_f(x) := \exp\left(\sum_{p \leq x} \frac{f(p)}{p}\right).$$

The shifted-prime divisor function over shifted primes

For any $b \in \mathbb{Z} \setminus \{0\}$, let $N_b(x, y) := \#\{b < q \leq x : \omega^*(q-b) \geq y\}$.

Theorem 8 (F., 2025)

Fixing any $b \in \mathbb{Z} \setminus 2\mathbb{Z}$, there are constants $c_1, c_2 > 0$ such that

$$\frac{\pi(x)}{y^{c_1 \log\log y}} < N_b(x, y) \ll \frac{\pi(x) \log y}{y}$$

for all sufficiently large x and $y \leq x^{c_2/\log\log x}$.

The proof of Theorem 8 uses a slightly upgraded version of the result of Adleman, Pomerance and Rumely (1983) on the maximal order of ω^* and a bound on the number of shifted primes with a large shifted prime divisor due to Luca, Pizarro-Madariaga, and Pomerance (2014).

Shifted primes possessing a large shifted-prime divisor

McNew, Pollack and Pomerance (2017) showed that the number of $n \le x$ with a shifted prime divisor $q-1>y$ is

$$\ll \frac{x}{(\log y)^{\eta}\sqrt{\log\log y}},$$

where

$$\eta := 1 - \frac{1+\log\log 2}{\log 2} = 0.0860713...$$

is the Erdős–Tenenbaum–Ford constant. This makes more precise the bound $\ll x/(\log y)^c$ for some $c>0$ due to Erdős and Wagstaff (1980). Ford (2017) further refined these results for y in various ranges.

Luca, Pizarro-Madariaga, Pomerance (2014) studied the shifted-prime analogue, proving that for any $u \in \mathbb{N}$ and $v \in \mathbb{Z}$, there is a constant $c = c(u,v) > 0$ such that for $x, y \ge 3$, the number of $p \le x$ such that $up+v$ has a shifted-prime divisor $q-1>y$ with $q \ne p$ is

$$\ll_{u,v} \frac{\pi(x)}{(\log y)^c}.$$

Shifted primes possessing a large shifted-prime divisor

Theorem 9 (F., 2025+)

Let $a \in \mathbb{Z} \setminus \{0\}$, $u \in \mathbb{N}$ *and* $v \in \mathbb{Z} \setminus \{-au\}$. *The number of primes* $p \leq x$ *such that* $up + v$ *has a shifted-prime divisor* $q - a > y$ *is*

$$\ll_{a,u,v} \frac{\pi(x)}{(\log y)^{\eta}\sqrt{\log\log y}}$$

for all $x, y \geq 3$. *In addition, if* $3 \leq y \leq x/2$, *then the count is*

$$\ll_{a,u,v} \frac{\pi(x)\log(x/y)}{\log x}.$$

The proof of Theorem 9 is based on a general Hardy–Ramanujan type inequality for the count of numbers in a sifted set with a prescribed number of prime factors (F., 2025+).

A Hardy–Ramanujan inequality for sifted sets

Let $f\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a "nice" multiplicative function, and let $\mathcal{S}$ be the set of integers $n \in [1, x]$ which avoid a subset $\mathcal{E}_p \subseteq (\mathbb{Z}/p\mathbb{Z})^\times$ of $\nu(p) \ll 1$ reduced residue classes modulo p for every prime p. Then

$$\sum_{\substack{n \in \mathcal{S} \\ \omega/\Omega(n) = k}} f(n) \ll \frac{x M_f(x)^{k-1}}{(k-1)! \log x} \prod_{p \leq x} \left(1 - \frac{\nu(p)}{p}\right)$$

uniformly for $1 \leq k \leq c_0 M_f(x)$, where $c_0 > 0$ is a suitable constant, and

$$M_f(x) := \sum_{p \leq x} \frac{f(p)}{p}.$$

Pollack (2020) proved similar results for ω and all $k \in \mathbb{N}$ when $\mathcal{S} = [1, x]$.

Shifted primes in the image of λ

Recall the Carmichael λ-function $\lambda(n) = \mathrm{Exp}(\mathbb{Z}/n\mathbb{Z})^\times$:

$$\lambda(p^v) = \begin{cases} \frac{1}{2}\varphi(p^v), & \text{if } p = 2 \text{ and } v \geq 3, \\ \varphi(p^v), & \text{otherwise,} \end{cases}$$

and $\lambda(n) = \mathrm{lcm}\{\lambda(p^v) \colon p^v \parallel n\}$.

Luca and Pomerance (2013) proved that

$$\#(\lambda(\mathbb{N}) \cap [1, x]) \leq \frac{x}{(\log x)^{\eta + o(1)}},$$

and the order matching lower bound was furnished by Ford, Luca, and Pomerance (2014).

Shifted primes in the image of λ

Corollary 10

Given any $u \in \mathbb{N}$ *and* $v \in \mathbb{Z} \setminus \{-u\}$, *we have*

$$\#((u\mathbb{P}+v) \cap \lambda(\mathbb{N}) \cap [1,x]) \le \frac{\pi(x)}{(\log x)^{\eta + o_{u,v}(1)}}$$

for $x \ge 3$. *On the other hand, we have*

$$\#((u\mathbb{P}-u) \cap \lambda(\mathbb{N}) \cap [1,x]) \asymp_u \pi(x)$$

for sufficiently large x.

It seems natural to conjecture that for any fixed $u \in \mathbb{N}$ and $v \in \mathbb{Z} \setminus \{-u\}$,

$$\#((u\mathbb{P}+v) \cap \lambda(\mathbb{N}) \cap [1,x]) = \frac{\pi(x)}{(\log x)^{\eta + o(1)}}.$$

Happy Birthday, Carl and Melvyn!